IDENTITIES OF SYMMETRY FOR GENERALIZED TWISTED BERNOULLI POLYNOMIALS TWISTED BY UNRAMIFIED ROOTS OF UNITY

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ABSTRACT. We derive three identities of symmetry in two variables and eight in three variables related to generalized twisted Bernoulli polynomials and generalized twisted power sums, both of which are twisted by unramified roots of unity. The case of ramified roots of unity was treated previously. The derivations of identities are based on the p-adic integral expression, with respect to a measure introduced by Koblitz, of the generating function for the generalized twisted Bernoulli polynomials and the quotient of p-adic integrals that can be expressed as the exponential generating function for the generalized twisted power sums.

1. INTRODUCTION AND PRELIMINARIES

Let p be a fixed prime. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Assume that $|\cdot|_p$ is the normalized absolute value of $\mathbb{C}_p$, such that $|p|_p = \frac{1}{p}$. The group Γ of all roots of unity of $\mathbb{C}_p$ is the direct product of its subgroups Γ_u (the subgroup of unramified roots of unity) and Γ_r (the subgroup of ramified roots of unity). Namely,

$$\Gamma = \Gamma_u \cdot \Gamma_r, \quad \Gamma_u \cap \Gamma_r = \{1\},$$

where

$$\Gamma_u = \{\xi \in \mathbb{C}_p |\ \xi^r = 1 \text{ for some } r \in \mathbb{Z}_{>0} \text{ with } (r, p) = 1\},$$
$$\Gamma_r = \{\xi \in \mathbb{C}_p |\ \xi^{p^s} = 1 \text{ for some } s \in \mathbb{Z}_{>0}\}.$$

Let d be a fixed positive integer. Then we let

$$X = X_d = \varprojlim_{N} \mathbb{Z}/dp^N\mathbb{Z} = \cup_{a=0}^{dp-1}(a + dp\mathbb{Z}_p),$$

with

$$a + dp^N\mathbb{Z}_p = \{x \in X |\ x \equiv a \ (\text{mod } dp^N)\},$$

and let $\pi : X \to \mathbb{Z}_p$ be the map given by the inverse limit of the natural maps

$$\mathbb{Z}/dP^N\mathbb{Z} \to \mathbb{Z}/p^N\mathbb{Z}.$$

If g is a function on $\mathbb{Z}_p$, we will use the same notation to denote the function $g \circ \pi$. Let $\chi : (\mathbb{Z}/d\mathbb{Z})^* \to \bar{\mathbb{Q}}^*$ be a (primitive) Dirichlet character of conductor d. Then it will be pulled back to X via the natural map $X \to \mathbb{Z}/d\mathbb{Z}$. Here we fix, once and for all, an imbedding $\bar{\mathbb{Q}} \to \mathbb{C}_p$, so that χ is regarded as a map of X to $\mathbb{C}_p$.

Let $z \in \mathbb{C}_p$ be such that $z^{dp^N} \neq 1$, for all N. Then we define

$$\mu_z(a + dp^N \mathbb{Z}_p) = \frac{z^a}{z^{dp^N} - 1}. \tag{1}$$

Observe that (1) is -1 times of the corresponding one in [7]. Then it is known(cf. [7]) that μ_z extends to a measure on X if and only if $z \in \{x \in \mathbb{C}_p \mid |x - 1|_p \geq 1\}$. So for any such a z, and any continuous $\mathbb{C}_p$-valued function f on X,

$$\int_X f(x) d\mu_z(x) = \lim_{N \to \infty} \sum_{a=0}^{dp^N - 1} f(a) \mu(a + dp^N \mathbb{Z}_p)$$

$$= \lim_{N \to \infty} \frac{1}{z^{dp^N} - 1} \sum_{a=0}^{dp^N - 1} f(a) z^a. \tag{2}$$

Throughout this paper, we let $\xi \neq 1$ be any fixed r-th root of 1, with $(r, pd) = 1$ (and hence $\xi \in \Gamma_u$), and let

$$E = \{t \in \mathbb{C}_p \mid |t|_p < p^{-\frac{1}{p-1}}\}. \tag{3}$$

Then u_ξ is a measure on X, and, for a positive integer w, μ_{ξ^w} is a measure provided that w is not divisible by r. For each fixed $t \in E$ the function e^{zt} is analytic on $\mathbb{Z}_p$ and hence considered as a function on X. Using the definition (2), we get the p-adic integral expression of the generating function for the generalized twisted Bernoulli numbers $B_{n,\chi,\xi}$ attached to χ and ξ;

$$t \int_X \chi(z) e^{zt} d\mu_\xi(z) = \frac{t}{\xi^d e^{dt} - 1} \sum_{a=0}^{d-1} \chi(a) \xi^a e^{at} = \sum_{n=0}^{\infty} B_{n,\chi,\xi} \frac{t^n}{n!} \quad (t \in E). \tag{4}$$

So we have the following p-adic integral expression of the generating function for the generalized twisted Bernoulli polynomials $B_{n,\chi,\xi}(x)$ attached to χ and ξ;

$$t \int_X \chi(z) e^{(x+z)t} d\mu_\xi(z) = \frac{t e^{xt}}{\xi^d e^{dt} - 1} \sum_{a=0}^{d-1} \chi(a) \xi^a e^{at}$$

$$= \sum_{n=0}^{\infty} B_{n,\chi,\xi}(x) \frac{t^n}{n!} \quad (t \in E, x \in \mathbb{Z}_p). \tag{5}$$

Also, from (2) we have the p-adic integral expression of the generating function for the twisted Bernoulli numbers $B_{n,\xi}$:

$$t \int_X e^{zt} d\mu_\xi(z) = \frac{t}{\xi e^t - 1} = \sum_{n=0}^{\infty} B_{n,\xi} \frac{t^n}{n!} \quad (t \in E). \tag{6}$$

Thus we obtain the p-adic integral expression of the generating function for the twisted Bernoulli polynomials $B_{n,\xi}(x)$:

$$t \int_X e^{(x+z)t} d\mu_\xi(z) = \frac{t e^{xt}}{\xi e^t - 1} = \sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!} \quad (t \in E, x \in \mathbb{Z}_p).$$

Let $S_k(n; \chi, \xi)$ denote the k-th generalized twisted power sum of the first $n + 1$ nonnegative integers attached to χ and ξ, namely

$$S_k(n; \chi, \xi) = \sum_{a=0}^{n} \chi(a) \xi^a a^k = \chi(0) \xi^0 0^k + \chi(1) \xi^1 1^k + \cdots + \chi(n) \xi^n n^k. \tag{7}$$

From (4), (6), and (7), one easily derives the following identities: for $w \in \mathbb{Z}_{>0}$, with w not divisible by r,

$$\frac{\int_X \chi(x)e^{xt}d\mu_\xi(x)}{\int_X e^{dwyt}d\mu_{\xi^{dw}}(y)} = \frac{\xi^{dw}e^{dwt}-1}{\xi^d e^{dt}-1}\sum_{a=0}^{d-1}\chi(a)\xi^a e^{at} \tag{8}$$

$$= \sum_{a=0}^{dw-1}\chi(a)\xi^a e^{at} \tag{9}$$

$$= \sum_{k=0}^{\infty} S_k(dw-1;\chi,\xi)\frac{t^k}{k!} \quad (t\in E). \tag{10}$$

In what follows, we will always assume that the p-adic integrals of the various (twisted) exponential functions on X are defined for $t \in E$ (cf. (3)), and therefore it will not be mentioned.

[1], [2], [5], [8], and [9] are some of the previous works on identities of symmetry in two variables involving Bernoulli polynomials and power sums. For the brief history, one is referred to those papers. For the first time, the idea of [5] was extended in [4] to the case of three variables so as to yield many new identities with abundant symmetry. This added some new identities of symmetry even to the existing ones in two variables as well.

On the other hand, in [6] the author obtained identities of symmetry in two variables involving generalized twisted Bernoulli polynomials and generalized twisted power sums, both of which are twisted by ramified roots of unity (i.e., p-power roots of unity). In [3], this was also extended to the case of three variables. In these ramified cases, p-adic Volkenborn-type integrals are used in both [3] and [6].

In this paper, in order to treat the unramified roots of unity case (i.e., the orders of the roots of unity are prime to p and the conductors of Dirichlet characters), we will adopt the measure introduced by Koblitz(cf. [7]) instead of Volkenborn measure, as stated in (1) and (2). It seems this idea has never been exploited before. In the end, we will be able to derive three identities of symmetry in two variables(cf. (44)-(46)) and eight in three variables related to generalized twisted Bernoulli polynomials and generalized twisted power sums (cf. (47)-(50), (53)-(56)).

The following is stated as Theorem 8 and an example of the full six symmetries in any positive integers w_1, w_2, w_3, with $w_1 w_2 w_3$ not divisible by r.

$$w_1^n \sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{dw_1-1}\chi(a)\xi^{aw_2w_3}B_{k,\chi,\xi^{aw_1w_3}}(w_2y_1+\frac{w_2}{w_1}a)S_{n-k}(dw_3-1;\chi,\xi^{w_1w_2})w_2^{n-k}w_3^k$$

$$= w_1^n \sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{dw_1-1}\chi(a)\xi^{aw_2w_3}B_{k,\chi,\xi^{w_1w_2}}(w_3y_1+\frac{w_3}{w_1}a)S_{n-k}(dw_2-1;\chi,\xi^{w_1w_3})w_3^{n-k}w_2^k$$

$$= w_2^n \sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{dw_2-1}\chi(a)\xi^{aw_1w_3}B_{k,\chi,\xi^{w_2w_3}}(w_1y_1+\frac{w_1}{w_2}a)S_{n-k}(dw_3-1;\chi,\xi^{w_1w_2})w_1^{n-k}w_3^k$$

$$= w_2^n \sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{dw_2-1}\chi(a)\xi^{aw_1w_3}B_{k,\chi,\xi^{w_1w_2}}(w_3y_1+\frac{w_3}{w_2}a)S_{n-k}(dw_1-1;\chi,\xi^{w_2w_3})w_3^{n-k}w_1^k$$

$$= w_3^n \sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{dw_3-1}\chi(a)\xi^{aw_1w_2}B_{k,\chi,\xi^{w_2w_3}}(w_1y_1+\frac{w_1}{w_3}a)S_{n-k}(dw_2-1;\chi,\xi^{w_1w_3})w_1^{n-k}w_2^k$$

$$= w_3^n \sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{dw_3-1}\chi(a)\xi^{aw_1w_2}B_{k,\chi,\xi^{w_1w_3}}(w_2y_1+\frac{w_2}{w_3}a)S_{n-k}(dw_1-1;\chi,\xi^{w_2w_3})w_2^{n-k}w_1^k.$$

The derivations of identities are based on the p-adic integral expression of the generating function for the generalized twisted Bernoulli polynomials in (5) and the

quotient of integrals in (8)-(10) that can be expressed as the exponential generating function for the generalized twisted power sums. These abundance of symmetries would not be unearthed if such p-adic integral representations had not been available. We indebted this idea to the paper [5].

2. Several types of quotients of p-adic integrals in two variables

Here we will introduce several types of quotients of p-adic integrals on X or X^2 from which some interesting identities follow owing to the built-in symmetries in w_1, w_2. In the following, w_1, w_2 are positive integers with suitable restrictions and the explicit expression of integral in (11) is obtained from the identities in (4) and (6).

(α) Type Γ^i (for $i = 0, 1, 2$)

$$I(\Gamma^i) = \frac{\int_{X^2} \chi(x_1)\chi(x_2) t^{2-i} e^{(w_1 x_1 + w_2 x_2 + w_1 w_2 (\sum_{j=1}^{2-i} y_j))t} d\mu_{\xi^{w_1}}(x_1) d\mu_{\xi^{w_2}}(x_2)}{\left(\int_X e^{dw_1 w_2 x_3 t} d\mu_{\xi^{dw_1 w_2}}(x_3)\right)^i} \tag{11}$$

$$= \frac{t^{2-i} e^{w_1 w_2 (\sum_{j=1}^{2-i} y_j)t} (\xi^{dw_1 w_2} e^{dw_1 w_2 t} - 1)^i (\sum_{a=0}^{d-1} \chi(a)\xi^{aw_1} e^{aw_1 t})(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_2} e^{aw_2 t})}{(\xi^{dw_1} e^{dw_1 t} - 1)(\xi^{dw_2} e^{dw_2 t} - 1)}. \tag{12}$$

The above p-adic integral is invariant under the transposition of w_1, w_2 as one can see either from the p-adic integral representation in (11) or from its explicit evaluation in (12).

3. Several types of quotients of p-adic integrals in three variables

Here we will introduce several types of quotients of p-adic integrals on X or X^3 from which some interesting identities follow due to the built-in symmetries in w_1, w_2, w_3. In the following, w_1, w_2, w_3 are positive integers with suitable restrictions and all of the explicit expressions of integrals in (14), (16), (18), and (20) are obtained from the identities in (4) and (6).

(a) Type Λ_{23}^i (for $i = 0, 1, 2, 3$)

$$I(\Lambda_{23}^i) = \frac{\int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3) t^{3-i} e^{(w_2 w_3 x_1 + w_1 w_3 x_2 + w_1 w_2 x_3 + w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j))t} \times d\mu_{\xi^{w_2 w_3}}(x_1) d\mu_{\xi^{w_1 w_3}}(x_2) d\mu_{\xi^{w_1 w_2}}(x_3)}{\left(\int_X e^{dw_1 w_2 w_3 x_4 t} d\mu_{\xi^{dw_1 w_2 w_3}}(x_4)\right)^i} \tag{13}$$

$$= \frac{t^{3-i} e^{w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j)t} (\xi^{dw_1 w_2 w_3} e^{dw_1 w_2 w_3 t} - 1)^i}{(\xi^{dw_2 w_3} e^{dw_2 w_3 t} - 1)(\xi^{dw_1 w_3} e^{dw_1 w_3 t} - 1)(\xi^{dw_1 w_2} e^{dw_1 w_2 t} - 1)}$$

$$\times \left(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_2 w_3} e^{aw_2 w_3 t}\right) \left(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_1 w_3} e^{aw_1 w_3 t}\right) \left(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_1 w_2} e^{aw_1 w_2 t}\right). \tag{14}$$

Here $w_2 w_3, w_1 w_3, w_1 w_2$ are not divisible by r, for $i = 0$, and $w_1 w_2 w_3$ is not divisible by r, for $i = 1, 2, 3$.

(b) Type Λ_{13}^i (for $i = 0, 1, 2, 3$)

$$I(\Lambda_{13}^i) = \frac{\int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3) t^{3-i} e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j))t} d\mu_{\xi^{w_1}}(x_1) \times d\mu_{\xi^{w_2}}(x_2) d\mu_{\xi^{w_3}}(x_3)}{(\int_X e^{dw_1 w_2 w_3 x_4 t} d\mu_{\xi^{dw_1 w_2 w_3}}(x_4))^i} \tag{15}$$

$$= \frac{t^{3-i} e^{w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j)t} (\xi^{dw_1 w_2 w_3} e^{dw_1 w_2 w_3 t} - 1)^i}{(\xi^{dw_1} e^{dw_1 t} - 1)(\xi^{dw_2} e^{dw_2 t} - 1)(\xi^{dw_3} e^{dw_3 t} - 1)}$$

$$\times \left(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_1} e^{aw_1 t} \right) \left(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_2} e^{aw_2 t} \right) \left(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_3} e^{aw_3 t} \right).$$

(16)

Here w_1, w_2, w_3 are not divisible by r, for $i = 0$, and $w_1 w_2 w_3$ is not divisible by r, for $i = 1, 2, 3$.

(c-0) Type Λ_{12}^0

$$I(\Lambda_{12}^0) = \int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3)t^3 e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_2 w_3 y + w_1 w_3 y + w_1 w_2 y)t}$$
$$\times d\mu_{\xi^{w_1}}(x_1)d\mu_{\xi^{w_2}}(x_2)d\mu_{\xi^{w_3}}(x_3) \qquad (17)$$
$$= \frac{t^3 e^{(w_2 w_3 + w_1 w_3 + w_1 w_2)yt}(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_1} e^{aw_1 t})(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_2} e^{aw_2 t})(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_3} e^{aw_3 t})}{(\xi^{dw_1} e^{dw_1 t} - 1)(\xi^{dw_2} e^{dw_2 t} - 1)(\xi^{dw_3} e^{dw_3 t} - 1)}.$$

(18)

Here w_1, w_2, w_3 are not divisible by r.

(c-1) Type Λ_{12}^1

$$I(\Lambda_{12}^1) = \frac{\int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3)e^{(w_1 x_1 + w_2 x_2 + w_3 x_3)t}d\mu_{\xi^{w_1}}(x_1)d\mu_{\xi^{w_2}}(x_2)d\mu_{\xi^{w_3}}(x_3)}{\int_X e^{dw_2 w_3 z_1 t}d\mu_{\xi^{dw_2 w_3}}(z_1)\int_X e^{dw_1 w_3 z_2 t}d\mu_{\xi^{dw_1 w_3}}(z_2)\int_X e^{dw_1 w_2 z_3 t}d\mu_{\xi^{dw_1 w_2}}(z_3)} \qquad (19)$$
$$= \frac{(\xi^{dw_2 w_3} e^{dw_2 w_3 t} - 1)(\xi^{dw_1 w_3} e^{dw_1 w_3 t} - 1)(\xi^{dw_1 w_2} e^{dw_1 w_2 t} - 1)}{(\xi^{dw_1} e^{dw_1 t} - 1)(\xi^{dw_2} e^{dw_2 t} - 1)(\xi^{dw_3} e^{dw_3 t} - 1)}$$
$$\times \left(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_1} e^{aw_1 t} \right) \left(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_2} e^{aw_2 t} \right) \left(\sum_{a=0}^{d-1} \chi(a)\xi^{aw_3} e^{aw_3 t} \right). \qquad (20)$$

Here $w_2 w_3, w_1 w_3, w_1 w_2$ are not divisible by r.

All of the above p-adic integrals of various types are invariant under all permutations of w_1, w_2, w_3, as one can see either from p-adic integral representations in (13), (15), (17), and (19) or their explicit evaluations in (14), (16), (18), and (20).

4. Identities for generalized twisted Bernoulli polynomials in two variables

All of the following results can be easily obtained from (5) and (8)-(10).

(α-0)

$$I(\Gamma^0) = \int_X \chi(x_1)te^{w_1(x_1 + w_2 y_1)t}d\mu_{\xi^{w_1}}(x_1) \int_X \chi(x_2)te^{w_2(x_2 + w_1 y_2)t}d\mu_{\xi^{w_2}}(x_2)$$
$$= \left(\sum_{k=0}^{\infty} B_{k,\chi,\xi^{w_1}}(w_2 y_1)\frac{(w_1 t)^k}{k!} \right) \left(\sum_{l=0}^{\infty} B_{l,\chi,\xi^{w_2}}(w_1 y_2)\frac{(w_2 t)^l}{l!} \right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_1}}(w_2 y_1)B_{n-k,\chi,\xi^{w_2}}(w_1 y_2)w_1^k w_2^{n-k} \right) \frac{t^n}{n!}. \qquad (21)$$

(α-1) Here we write $I(\Gamma^1)$ in two different ways:

(1)

$$I(\Gamma^1) = \int_X \chi(x_1)te^{w_1(x_1 + w_2 y_1)t}d\mu_{\xi^{w_1}}(x_1) \times \frac{\int_X \chi(x_2)e^{w_2 x_2 t}d\mu_{\xi^{w_2}}(x_2)}{\int_X e^{dw_1 w_2 x_3 t}d\mu_{\xi^{dw_1 w_2}}(x_3)} \qquad (22)$$
$$= \left(\sum_{k=0}^{\infty} B_{k,\chi,\xi^{w_1}}(w_2 y_1)\frac{(w_1 t)^k}{k!} \right) \left(\sum_{l=0}^{\infty} S_l(dw_1 - 1; \chi, \xi^{w_2})\frac{(w_2 t)^l}{l!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_1}}(w_2 y_1) S_{n-k}(dw_1 - 1; \chi, \xi^{w_2}) w_1^k w_2^{n-k} \right) \frac{t^n}{n!}. \qquad (23)$$

(2) Invoking (9), (22) can also be written as

$$
\begin{aligned}
I(\Gamma^1) &= \sum_{a=0}^{dw_1-1} \chi(a) \xi^{aw_2} \int_X \chi(x_1) t e^{w_1(x_1 + w_2 y_1 + \frac{w_2}{w_1}a)t} d\mu_{\xi^{w_1}}(x_1) \\
&= \sum_{a=0}^{dw_1-1} \chi(a) \xi^{aw_2} \left(\sum_{n=0}^{\infty} B_{n,\chi,\xi^{w_1}}\left(w_2 y_1 + \frac{w_2}{w_1}a\right) \frac{(w_1 t)^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(w_1^n \sum_{a=0}^{dw_1-1} \chi(a) \xi^{aw_2} B_{n,\chi,\xi^{w_1}}\left(w_2 y_1 + \frac{w_2}{w_1}a\right) \right) \frac{t^n}{n!}. \qquad (24)
\end{aligned}
$$

(α-2)

$$
\begin{aligned}
I(\Gamma^2) &= \frac{\int_X \chi(x_1) e^{w_1 x_1 t} d\mu_{\xi^{w_1}}(x_1)}{\int_X e^{dw_1 w_2 x_3 t} d\mu_{\xi^{dw_1 w_2}}(x_3)} \times \frac{\int_X \chi(x_2) e^{w_2 x_2 t} d\mu_{\xi^{w_2}}(x_2)}{\int_X e^{dw_1 w_2 x_3 t} d\mu_{\xi^{dw_1 w_2}}(x_3)} \\
&= \left(\sum_{k=0}^{\infty} S_k(dw_2 - 1; \chi, \xi^{w_1}) \frac{(w_1 t)^k}{k!} \right) \left(\sum_{l=0}^{\infty} S_l(dw_1 - 1; \chi, \xi^{w_2}) \frac{(w_2 t)^l}{l!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} S_k(dw_2 - 1; \chi, \xi^{w_1}) S_{n-k}(dw_1 - 1; \chi, \xi^{w_2}) w_1^k w_2^{n-k} \right) \frac{t^n}{n!}.
\end{aligned}
$$

5. Identities for generalized twisted Bernoulli polynomials in three variables

All of the following results can be easily obtained from (5) and (8)-(10). First, let's consider Type Λ_{23}^i, for each $i = 0, 1, 2, 3$.

(a-0)

$$
\begin{aligned}
I(\Lambda_{23}^0) &= \int_X \chi(x_1) t e^{w_2 w_3(x_1 + w_1 y_1)t} d\mu_{\xi^{w_2 w_3}}(x_1) \int_X \chi(x_2) t e^{w_1 w_3(x_2 + w_2 y_2)t} d\mu_{\xi^{w_1 w_3}}(x_2) \\
&\qquad \times \int_X \chi(x_3) t e^{w_1 w_2(x_3 + w_3 y_3)t} d\mu_{\xi^{w_1 w_2}}(x_3) \\
&= \left(\sum_{k=0}^{\infty} \frac{B_{k,\chi,\xi^{w_2 w_3}}(w_1 y_1)}{k!}(w_2 w_3 t)^k \right) \left(\sum_{l=0}^{\infty} \frac{B_{l,\chi,\xi^{w_1 w_3}}(w_2 y_2)}{l!}(w_1 w_3 t)^l \right) \\
&\qquad \times \left(\sum_{m=0}^{\infty} \frac{B_{m,\chi,\xi^{w_1 w_2}}(w_3 y_3)}{m!}(w_1 w_2 t)^m \right) \\
&= \sum_{n=0}^{\infty} (\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_2 w_3}}(w_1 y_1) B_{l,\chi,\xi^{w_1 w_3}}(w_2 y_2) B_{m,\chi,\xi^{w_1 w_2}}(w_3 y_3) \\
&\qquad \times w_1^{l+m} w_2^{k+m} w_3^{k+l}) \frac{t^n}{n!}, \qquad (25)
\end{aligned}
$$

where the inner sum is over all nonnegative integers k, l, m, with $k+l+m = n$ and

$$\binom{n}{k,l,m} = \frac{n!}{k!\, l!\, m!}.$$

(a-1) Here we write $I(\Lambda_{23}^1)$ in two different ways:

(1)

$$I(\Lambda_{23}^1) = \int_X \chi(x_1) t e^{w_2 w_3 (x_1 + w_1 y_1) t} d\mu_{\xi^{w_2 w_3}}(x_1) \int_X \chi(x_2) t e^{w_1 w_3 (x_2 + w_2 y_2) t} d\mu_{\xi^{w_1 w_3}}(x_2)$$

$$\times \frac{\int_X \chi(x_3) e^{w_1 w_2 x_3 t} d\mu_{\xi^{w_1 w_2}}(x_3)}{\int_X e^{d w_1 w_2 w_3 x_4 t} d\mu_{\xi^{d w_1 w_2 w_3}}(x_4)} \qquad (26)$$

$$= \left(\sum_{k=0}^{\infty} B_{k,\chi,\xi^{w_2 w_3}}(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!} \right) \left(\sum_{l=0}^{\infty} B_{l,\chi,\xi^{w_1 w_3}}(w_2 y_2) \frac{(w_1 w_3 t)^l}{l!} \right)$$

$$\times \left(\sum_{m=0}^{\infty} S_m(d w_3 - 1; \chi, \xi^{w_1 w_2}) \frac{(w_1 w_2 t)^m}{m!} \right)$$

$$= \sum_{n=0}^{\infty} \Big(\sum_{k+l+m=n} \binom{n}{k,\, l,\, m} B_{k,\chi,\xi^{w_2 w_3}}(w_1 y_1) B_{l,\chi,\xi^{w_1 w_3}}(w_2 y_2) S_m(d w_3 - 1; \chi, \xi^{w_1 w_2})$$

$$\times w_1^{l+m} w_2^{k+m} w_3^{k+l} \Big) \frac{t^n}{n!}. \qquad (27)$$

(2) Invoking (9), (20) can also be written as

$$I(\Lambda_{23}^1) = \sum_{a=0}^{d w_3 - 1} \chi(a) \xi^{a w_1 w_2} \int_X \chi(x_1) t e^{w_2 w_3 (x_1 + w_1 y_1) t} d\mu_{\xi^{w_2 w_3}}(x_1)$$

$$\times \int_X \chi(x_2) t e^{w_1 w_3 (x_2 + w_2 y_2 + \frac{w_2}{w_3} a) t} d\mu_{\xi^{w_1 w_3}}(x_2)$$

$$= \sum_{a=0}^{d w_3 - 1} \chi(a) \xi^{a w_1 w_2} \left(\sum_{k=0}^{\infty} B_{k,\chi,\xi^{w_2 w_3}}(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!} \right)$$

$$\times \left(\sum_{l=0}^{\infty} B_{l,\chi,\xi^{w_1 w_3}}(w_2 y_2 + \frac{w_2}{w_3} a) \frac{(w_1 w_3 t)^l}{l!} \right)$$

$$= \sum_{n=0}^{\infty} \Big(w_3^n \sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_2 w_3}}(w_1 y_1) \sum_{a=0}^{d w_3 - 1} \chi(a) \xi^{a w_1 w_2}$$

$$\times B_{n-k,\chi,\xi^{w_1 w_3}}(w_2 y_2 + \frac{w_2}{w_3} a) w_1^{n-k} w_2^k \Big) \frac{t^n}{n!}. \qquad (28)$$

(a-2) Here we write $I(\Lambda_{23}^2)$ in three different ways:

(1)

$$I(\Lambda_{23}^2) = \int_X \chi(x_1) t e^{w_2 w_3 (x_1 + w_1 y_1) t} d\mu_{\xi^{w_2 w_3}}(x_1)$$

$$\times \frac{\int_X \chi(x_2) e^{w_1 w_3 x_2 t} d\mu_{\xi^{w_1 w_3}}(x_2)}{\int_X e^{d w_1 w_2 w_3 x_4 t} d\mu_{\xi^{d w_1 w_2 w_3}}(x_4)} \times \frac{\int_X \chi(x_3) e^{w_1 w_2 x_3 t} d\mu_{\xi^{w_1 w_2}}(x_3)}{\int_X e^{d w_1 w_2 w_3 x_4 t} d\mu_{\xi^{d w_1 w_2 w_3}}(x_4)} \qquad (29)$$

$$= \left(\sum_{k=0}^{\infty} B_{k,\chi,\xi^{w_2 w_3}}(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!} \right) \left(\sum_{l=0}^{\infty} S_l(d w_2 - 1; \chi, \xi^{w_1 w_3}) \frac{(w_1 w_3 t)^l}{l!} \right)$$

$$\times \left(\sum_{m=0}^{\infty} S_m(d w_3 - 1; \chi, \xi^{w_1 w_2}) \frac{(w_1 w_2 t)^m}{m!} \right)$$

$$= \sum_{n=0}^{\infty} \Big(\sum_{k+l+m=n} \binom{n}{k,\, l,\, m} B_{k,\chi,\xi^{w_2 w_3}}(w_1 y_1) S_l(d w_2 - 1; \chi, \xi^{w_1 w_3}) S_m(d w_3 - 1; \chi, \xi^{w_1 w_2})$$

$$\times w_1^{l+m} w_2^{k+m} w_3^{k+l} \Big) \frac{t^n}{n!}. \qquad (30)$$

(2) Invoking (9), (29) can also be written as

$$I(\Lambda_{23}^2) = \sum_{a=0}^{dw_2-1} \chi(a)\xi^{aw_1w_3} \int_X \chi(x_1)te^{w_2w_3(x_1+w_1y_1+\frac{w_1}{w_2}a)t}d\mu_{\xi^{w_2w_3}}(x_1)$$

$$\times \frac{\int_X \chi(x_3)e^{w_1w_2x_3t}d\mu_{\xi^{w_1w_2}}(x_3)}{\int_X e^{dw_1w_2w_3x_4t}d\mu_{\xi^{dw_1w_2w_3}}(x_4)} \qquad (31)$$

$$= \sum_{a=0}^{dw_2-1} \chi(a)\xi^{aw_1w_3} \left(\sum_{k=0}^{\infty} B_{k,\chi,\xi^{w_2w_3}}(w_1y_1 + \frac{w_1}{w_2}a)\frac{(w_2w_3t)^k}{k!}\right)$$

$$\times \left(\sum_{l=0}^{\infty} S_l(dw_3-1;\chi,\xi^{w_1w_2})\frac{(w_1w_2t)^l}{l!}\right)$$

$$= \sum_{n=0}^{\infty} (w_2^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{dw_2-1} \chi(a)\xi^{aw_1w_3} B_{k,\chi,\xi^{w_2w_3}}(w_1y_1 + \frac{w_1}{w_2}a)$$

$$\times S_{n-k}(dw_3-1;\chi,\xi^{w_1w_2})w_1^{n-k}w_3^k)\frac{t^n}{n!}. \qquad (32)$$

(3) Invoking (9) once again, (31) can be written as

$$I(\Lambda_{23}^2) = \sum_{a=0}^{dw_2-1} \chi(a)\xi^{aw_1w_3} \sum_{b=0}^{dw_3-1} \chi(b)\xi^{bw_1w_2} \int_X \chi(x_1)te^{w_2w_3(x_1+w_1y_1+\frac{w_1}{w_2}a+\frac{w_1}{w_3}b)t}d\mu_{\xi^{w_2w_3}}(x_1)$$

$$= \sum_{a=0}^{dw_2-1} \chi(a)\xi^{aw_1w_3} \sum_{b=0}^{dw_3-1} \chi(b)\xi^{bw_1w_2} \left(\sum_{n=0}^{\infty} B_{n,\chi,\xi^{w_2w_3}}(w_1y_1 + \frac{w_1}{w_2}a + \frac{w_1}{w_3}b)\frac{(w_2w_3t)^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \left((w_2w_3)^n \sum_{a=0}^{dw_2-1} \sum_{b=0}^{dw_3-1} \chi(ab)\xi^{w_1(aw_3+bw_2)} B_{n,\chi,\xi^{w_2w_3}}(w_1y_1 + \frac{w_1}{w_2}a + \frac{w_1}{w_3}b)\right)\frac{t^n}{n!}. \qquad (33)$$

(a-3)

$$I(\Lambda_{23}^3) = \frac{\int_X \chi(x_1)e^{w_2w_3x_1t}d\mu_{\xi^{w_2w_3}}(x_1)}{\int_X e^{dw_1w_2w_3x_4t}d\mu_{\xi^{dw_1w_2w_3}}(x_4)} \times \frac{\int_X \chi(x_2)e^{w_1w_3x_2t}d\mu_{\xi^{w_1w_3}}(x_2)}{\int_X e^{dw_1w_2w_3x_4t}d\mu_{\xi^{dw_1w_2w_3}}(x_4)}$$

$$\times \frac{\int_X \chi(x_3)e^{w_1w_2x_3t}d\mu_{\xi^{w_1w_2}}(x_3)}{\int_X e^{dw_1w_2w_3x_4t}d\mu_{\xi^{dw_1w_2w_3}}(x_4)}$$

$$= \left(\sum_{k=0}^{\infty} S_k(dw_1-1;\chi,\xi^{w_2w_3})\frac{(w_2w_3t)^k}{k!}\right) \left(\sum_{l=0}^{\infty} S_l(dw_2-1;\chi,\xi^{w_1w_3})\frac{(w_1w_3t)^l}{l!}\right)$$

$$\times \left(\sum_{m=0}^{\infty} S_m(dw_3-1;\chi,\xi^{w_1w_2})\frac{(w_1w_2t)^m}{m!}\right)$$

$$= \sum_{n=0}^{\infty} (\sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_1-1;\chi,\xi^{w_2w_3})S_l(dw_2-1;\chi,\xi^{w_1w_3})$$

$$\times S_m(dw_3-1;\chi,\xi^{w_1w_2})w_1^{l+m}w_2^{k+m}w_3^{k+l})\frac{t^n}{n!}. \qquad (34)$$

(b) For Type Λ_{13}^i $(i=0,1,2,3)$, we may consider the analogous things to the ones in (a-0), (a-1), (a-2), and (a-3). However, each of those can be obtained from the corresponding ones in (a-0), (a-1), (a-2), and (a-3). Indeed, if we substitute w_2w_3, w_1w_3, w_1w_2 respectively for w_1, w_2, w_3 in $\frac{I(\Lambda_{23}^i)}{t^{3-i}}$ (cf. (13)), this amounts to replacing t by $w_1w_2w_3t$ and ξ by $\xi^{w_1w_2w_3}$ in $\frac{I(\Lambda_{13}^i)}{t^{3-i}}$ (cf. (15)). So, upon replacing w_1, w_2, w_3 respectively by w_2w_3, w_1w_3, w_1w_2, dividing by $(w_1w_2w_3)^n$, and replacing $\xi^{w_1w_2w_3}$ by ξ, in each of the expressions of (25), (27), (28), (30), (32)-(34), we will get the corresponding symmetric identities for Type Λ_{13}^i $(i=0,1,2,3)$.

(c-0)

$$I(\Lambda_{12}^0) = \int_X \chi(x_1) t e^{w_1(x_1+w_2 y)t} d\mu_{\xi^{w_1}}(x_1) \int_X \chi(x_2) t e^{w_2(x_2+w_3 y)t} d\mu_{\xi^{w_2}}(x_2)$$

$$\times \int_X \chi(x_3) t e^{w_3(x_3+w_1 y)t} d\mu_{\xi^{w_3}}(x_3)$$

$$= \left(\sum_{k=0}^{\infty} \frac{B_{k,\chi,\xi^{w_1}}(w_2 y)}{k!}(w_1 t)^k \right) \left(\sum_{l=0}^{\infty} \frac{B_{l,\chi,\xi^{w_2}}(w_3 y)}{l!}(w_2 t)^l \right) \left(\sum_{m=0}^{\infty} \frac{B_{m,\chi,\xi^{w_3}}(w_1 y)}{m!}(w_3 t)^m \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1}}(w_2 y) B_{l,\chi,\xi^{w_2}}(w_3 y) B_{m,\chi,\xi^{w_3}}(w_1 y) w_1^k w_2^l w_3^m \right) \frac{t^n}{n!}. \quad (35)$$

(c-1)

$$I(\Lambda_{12}^1) = \frac{\int_X \chi(x_1) e^{w_1 x_1 t} d\mu_{\xi^{w_1}}(x_1)}{\int_X e^{dw_1 w_2 z_3 t} d\mu_{\xi^{dw_1 w_2}}(z_3)} \times \frac{\int_X \chi(x_2) e^{w_2 x_2 t} d\mu_{\xi^{w_2}}(x_2)}{\int_X e^{dw_2 w_3 z_1 t} d\mu_{\xi^{dw_2 w_3}}(z_1)} \times \frac{\int_X \chi(x_3) e^{w_3 x_3 t} d\mu_{\xi^{w_3}}(x_3)}{\int_X e^{dw_3 w_1 z_2 t} d\mu_{\xi^{dw_3 w_1}}(z_2)}$$

$$= \left(\sum_{k=0}^{\infty} S_k(dw_2 - 1; \chi, \xi^{w_1}) \frac{(w_1 t)^k}{k!} \right) \left(\sum_{l=0}^{\infty} S_l(dw_3 - 1; \chi, \xi^{w_2}) \frac{(w_2 t)^l}{l!} \right)$$

$$\times \left(\sum_{m=0}^{\infty} S_m(dw_1 - 1; \chi, \xi^{w_3}) \frac{(w_3 t)^m}{m!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_2 - 1; \chi, \xi^{w_1}) S_l(dw_3 - 1; \chi, \xi^{w_2}) \right.$$

$$\left. \times S_m(dw_1 - 1; \chi, \xi^{w_3}) w_1^k w_2^l w_3^m \right) \frac{t^n}{n!}. \quad (36)$$

6. Main theorems

As we noted earlier, the various types of quotients of p-adic integrals in Section 2 and Section 3 are respectively invariant under the transposition of w_1, w_2 and any permutation of w_1, w_2, w_3. So the corresponding expressions in Section 4 and Section 5 are respectively also invariant under the transposition of w_1, w_2 and any permutation of w_1, w_2, w_3. Thus our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in Section 5 yield distinct ones. In fact, as these expressions are obtained by permuting w_1, w_2, w_3 in a single one labeled by them, they can be viewed as a group in a natural manner and hence it is isomorphic to a quotient of S_3. In particular, the number of possible distinct expressions are 1,2,3, or 6. (a-0), (a-1(1)), (a-1(2)), and (a-2(2)) give the full six identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two identities of symmetry, while the expression in (a-3) yields no identities of symmetry. Similarly, (α-0), (α-1(1)), and (α-1(2)) give two identities of symmetry but (α-2) yields no identity of symmetry.

Here we will just consider the cases of Theorems 7 and 11, leaving the others as easy exercises for the reader. As for the case of Theorem 7, in addition to (50)-(52), we get the following three ones:

$$\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_2 w_3}}(w_1 y_1) S_l(dw_3 - 1; \chi, \xi^{w_1 w_2}) S_m(dw_2 - 1; \chi, \xi^{w_1 w_3}) w_1^{l+m} w_3^{k+m} w_2^{k+l}, \quad (37)$$

$$\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1 w_3}}(w_2 y_1) S_l(dw_1 - 1; \chi, \xi^{w_2 w_3}) S_m(dw_3 - 1; \chi, \xi^{w_1 w_2}) w_2^{l+m} w_1^{k+m} w_3^{k+l}, \quad (38)$$

$$\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1 w_2}}(w_3 y_1) S_l(dw_2 - 1; \chi, x^{w_1 w_3}) S_m(dw_1 - 1; \chi, \xi^{w_2 w_3}) w_3^{l+m} w_2^{k+m} w_1^{k+l}. \quad (39)$$

But, by interchanging l and m, we see that (37), (38), and (39) are respectively equal to (50), (51), and (52). As to Theorem 11, in addition to (56) and (57), we have:

$$\sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_2-1;\chi,\xi^{w_1})S_l(dw_3-1;\chi,\xi^{w_2})S_m(dw_1-1;\chi,\xi^{w_3})w_1^k w_2^l w_3^m, \quad (40)$$

$$\sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_3-1;\chi,\xi^{w_2})S_l(dw_1-1;\chi,\xi^{w_3})S_m(dw_2-1;\chi,\xi^{w_1})w_2^k w_3^l w_1^m, \quad (41)$$

$$\sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_3-1;\chi,\xi^{w_1})S_l(dw_2-1;\chi,\xi^{w_3})S_m(dw_1-1;\chi,\xi^{w_2})w_1^k w_3^l w_2^m, \quad (42)$$

$$\sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_2-1;\chi,\xi^{w_3})S_l(dw_1-1;\chi,\xi^{w_2})S_m(dw_3-1;\chi,\xi^{w_1})w_3^k w_2^l w_1^m. \quad (43)$$

However, (40) and (41) are equal to (56), as we can see by applying the permutations $k \to l, l \to m, m \to k$ for (40) and $k \to m, l \to k, m \to l$ for (41). Similarly, we see that (42) and (43) are equal to (57), by applying permutations $k \to l, l \to m, , m \to k$ for (42) and $k \to m, l \to k, m \to l$ for (43).

Theorem 1. *Let w_1, w_2 be any positive integers, such that r does not divide w_1, w_2. Then we have:*

$$\sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_1}}(w_2 y_1) B_{n-k,\chi,\xi^{w_2}}(w_1 y_2) w_1^k w_2^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_2}}(w_1 y_1) B_{n-k,\chi,\xi^{w_1}}(w_2 y_2) w_2^k w_1^{n-k}. \tag{44}$$

Theorem 2. *Let w_1, w_2 be any positive integers, such that r does not divide $w_1 w_2$. Then we have:*

$$\sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_1}}(w_2 y_1) S_{n-k}(dw_1-1;\chi,\xi^{w_2}) w_1^k w_2^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_2}}(w_1 y_1) S_{n-k}(dw_2-1;\chi,\xi^{w_1}) w_2^k w_1^{n-k}. \tag{45}$$

Theorem 3. *Let w_1, w_2 be any positive integers, such that r does not divide $w_1 w_2$. Then we have:*

$$w_1^n \sum_{a=0}^{dw_1-1} \chi(a)\xi^{aw_2} B_{n,\chi,\xi^{w_1}}\left(w_2 y_1 + \frac{w_2}{w_1}a\right)$$
$$= w_2^n \sum_{a=0}^{dw_2-1} \chi(a)\xi^{aw_1} B_{n,\chi,\xi^{w_2}}\left(w_1 y_1 + \frac{w_1}{w_2}a\right). \tag{46}$$

Theorem 4. *Let w_1, w_2, w_3 be any positive integers, such that r does not divide w_2w_3, w_1w_3, w_1w_2. Then we have:*

$$\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_2w_3}}(w_1y_1) B_{l,\chi,\xi^{w_1w_3}}(w_2y_2) B_{m,\chi,\xi^{w_1w_2}}(w_3y_3) w_1^{l+m} w_2^{k+m} w_3^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_2w_3}}(w_1y_1) B_{l,\chi,\xi^{w_1w_2}}(w_3y_2) B_{m,\chi,\xi^{w_1w_3}}(w_2y_3) w_1^{l+m} w_3^{k+m} w_2^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1w_3}}(w_2y_1) B_{l,\chi,\xi^{w_2w_3}}(w_1y_2) B_{m,\chi,\xi^{w_1w_2}}(w_3y_3) w_2^{l+m} w_1^{k+m} w_3^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1w_3}}(w_2y_1) B_{l,\chi,\xi^{w_1w_2}}(w_3y_2) B_{m,\chi,\xi^{w_2w_3}}(w_1y_3) w_2^{l+m} w_3^{k+m} w_1^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1w_2}}(w_3y_1) B_{l,\chi,\xi^{w_2w_3}}(w_1y_2) B_{m,\chi,\xi^{w_1w_3}}(w_2y_3) w_3^{l+m} w_1^{k+m} w_2^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1w_2}}(w_3y_1) B_{l,\chi,\xi^{w_1w_3}}(w_2y_2) B_{m,\chi,\xi^{w_2w_3}}(w_1y_3) w_3^{l+m} w_2^{k+m} w_1^{k+l}.$$

$$(47)$$

Theorem 5. *Let w_1, w_2, w_3 be any positive integers, such that r does not divide $w_1w_2w_3$. Then we have:*

$$\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_2w_3}}(w_1y_1) B_{l,\chi,\xi^{w_1w_3}}(w_2y_2) S_m(dw_3-1;\chi,\xi^{w_1w_2}) w_1^{l+m} w_2^{k+m} w_3^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_2w_3}}(w_1y_1) B_{l,\chi,\xi^{w_1w_2}}(w_3y_2) S_m(dw_2-1;\chi,\xi^{w_1w_3}) w_1^{l+m} w_3^{k+m} w_2^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1w_3}}(w_2y_1) B_{l,\chi,\xi^{w_2w_3}}(w_1y_2) S_m(dw_3-1;\chi,\xi^{w_1w_2}) w_2^{l+m} w_1^{k+m} w_3^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1w_3}}(w_2y_1) B_{l,\chi,\xi^{w_1w_2}}(w_3y_2) S_m(dw_1-1;\chi,\xi^{w_2w_3}) w_2^{l+m} w_3^{k+m} w_1^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1w_2}}(w_3y_1) B_{l,\chi,\xi^{w_1w_3}}(w_2y_2) S_m(dw_1-1;\chi,\xi^{w_2w_3}) w_3^{l+m} w_2^{k+m} w_1^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_1w_2}}(w_3y_1) B_{l,\chi,\xi^{w_2w_3}}(w_1y_2) S_m(dw_2-1;\chi,\xi^{w_1w_3}) w_3^{l+m} w_1^{k+m} w_2^{k+l}.$$

$$(48)$$

Theorem 6. *Let w_1, w_2, w_3 be any positive integers, such that r does not divide $w_1w_2w_3$. Then we have:*

$$w_1^n \sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_1w_2}}(w_3y_1) \sum_{a=0}^{dw_1-1} \chi(a)\xi^{aw_2w_3} B_{n-k,\chi,\xi^{w_1w_3}}\left(w_2y_2 + \frac{w_2}{w_1}a\right) w_3^{n-k} w_2^k$$

$$= w_1^n \sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_1w_3}}(w_2y_1) \sum_{a=0}^{dw_1-1} \chi(a)\xi^{aw_2w_3} B_{n-k,\chi,\xi^{w_1w_2}}\left(w_3y_2 + \frac{w_3}{w_1}a\right) w_2^{n-k} w_3^k$$

$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_1w_2}}(w_3y_1) \sum_{a=0}^{dw_2-1} \chi(a)\xi^{aw_1w_3} B_{n-k,\chi,\xi^{w_2w_3}}\left(w_1y_2 + \frac{w_1}{w_2}a\right) w_3^{n-k} w_1^k$$

$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_2w_3}}(w_1y_1) \sum_{a=0}^{dw_2-1} \chi(a)\xi^{aw_1w_3} B_{n-k,\chi,\xi^{w_1w_2}}\left(w_3y_2 + \frac{w_3}{w_2}a\right) w_1^{n-k} w_3^k$$

$$= w_3^n \sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_1w_3}}(w_2y_1) \sum_{i=0}^{dw_3-1} \chi(a)\xi^{aw_1w_2} B_{n-k,\chi,\xi^{w_2w_3}}\left(w_1y_2 + \frac{w_1}{w_3}a\right) w_2^{n-k} w_1^k$$

$$= w_3^n \sum_{k=0}^{n} \binom{n}{k} B_{k,\chi,\xi^{w_2w_3}}(w_1y_1) \sum_{a=0}^{dw_3-1} \chi(a)\xi^{aw_1w_2} B_{n-k,\chi,\xi^{w_1w_3}}\left(w_2y_2 + \frac{w_2}{w_3}a\right) w_1^{n-k} w_2^k$$

$$(49)$$

Theorem 7. *Let w_1, w_2, w_3 be any positive integers, such that r does not divide $w_1w_2w_3$. Then we have the following three symmetries in w_1, w_2, w_3:*

$$\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_2w_3}}(w_1y_1) S_l(dw_2-1;\chi,\xi^{w_1w_3}) S_m(dw_3-1;\chi,\xi^{w_1w_2}) w_1^{l+m} w_2^{k+m} w_3^{k+l} \quad (50)$$

$$= \sum_{k+l+m=n} \binom{n}{k,\,l,\,m} B_{k,\chi,\xi^{w_1 w_3}}(w_2 y_1) S_l(dw_3 - 1; \chi, \xi^{w_1 w_2}) S_m(dw_1 - 1; \chi, \xi^{w_2 w_3}) w_2^{l+m} w_3^{k+m} w_1^{k+l}$$

$$\tag{51}$$

$$= \sum_{k+l+m=n} \binom{n}{k,\,l,\,m} B_{k,\chi,\xi^{w_1 w_2}}(w_3 y_1) S_l(dw_1 - 1; \chi, \xi^{w_2 w_3}) S_m(dw_2 - 1; \chi, \xi^{w_1 w_3}) w_3^{l+m} w_1^{k+m} w_2^{k+l}.$$

$$\tag{52}$$

Theorem 8. *Let w_1, w_2, w_3 be any positive integers, such that r does not divide $w_1 w_2 w_3$. Then we have:*

$$w_1^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{dw_1-1} \chi(a) \xi^{aw_2 w_3} B_{k,\chi,\xi^{aw_1 w_3}}\left(w_2 y_1 + \frac{w_2}{w_1} a\right) S_{n-k}(dw_3 - 1; \chi, \xi^{w_1 w_2}) w_2^{n-k} w_3^{k}$$

$$= w_1^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{dw_1-1} \chi(a) \xi^{aw_2 w_3} B_{k,\chi,\xi^{w_1 w_2}}\left(w_3 y_1 + \frac{w_3}{w_1} a\right) S_{n-k}(dw_2 - 1; \chi, \xi^{w_1 w_3}) w_3^{n-k} w_2^{k}$$

$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{dw_2-1} \chi(a) \xi^{aw_1 w_3} B_{k,\chi,\xi^{w_2 w_3}}\left(w_1 y_1 + \frac{w_1}{w_2} a\right) S_{n-k}(dw_3 - 1; \chi, \xi^{w_1 w_2}) w_1^{n-k} w_3^{k}$$

$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{dw_2-1} \chi(a) \xi^{aw_1 w_3} B_{k,\chi,\xi^{w_1 w_2}}\left(w_3 y_1 + \frac{w_3}{w_2} a\right) S_{n-k}(dw_1 - 1; \chi, x^{w_2 w_3}) w_3^{n-k} w_1^{k}$$

$$= w_3^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{dw_3-1} \chi(a) \xi^{aw_1 w_2} B_{k,\chi,\xi^{w_2 w_3}}\left(w_1 y_1 + \frac{w_1}{w_3} a\right) S_{n-k}(dw_2 - 1; \chi, \xi^{w_1 w_3}) w_1^{n-k} w_2^{k}$$

$$= w_3^n \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{dw_3-1} \chi(a) \xi^{aw_1 w_2} B_{k,\chi,\xi^{w_1 w_3}}\left(w_2 y_1 + \frac{w_2}{w_3} a\right) S_{n-k}(dw_1 - 1; \chi, \xi^{w_2 w_3}) w_2^{n-k} w_1^{k}.$$

$$\tag{53}$$

Theorem 9. *Let w_1, w_2, w_3 be any positive integers, such that r does not divide $w_1 w_2 w_3$. Then we have the following three symmetries in w_1, w_2, w_3:*

$$(w_1 w_2)^n \sum_{a=0}^{dw_1-1} \sum_{b=0}^{dw_2-1} \chi(ab) \xi^{w_3(aw_2+bw_1)} B_{n,\chi,\xi^{w_1 w_2}}\left(w_3 y_1 + \frac{w_3}{w_1} a + \frac{w_3}{w_2} b\right)$$

$$= (w_2 w_3)^n \sum_{a=0}^{dw_2-1} \sum_{b=0}^{dw_3-1} \chi(ab) \xi^{w_1(aw_3+bw_2)} B_{n,\chi,\xi^{w_2 w_3}}\left(w_1 y_1 + \frac{w_1}{w_2} a + \frac{w_1}{w_3} b\right) \tag{54}$$

$$= (w_3 w_1)^n \sum_{a=0}^{dw_3-1} \sum_{b=0}^{dw_1-1} \chi(ab) \xi^{w_2(aw_1+bw_3)} B_{n,\chi,\xi^{w_1 w_3}}\left(w_2 y_1 + \frac{w_2}{w_3} a + \frac{w_2}{w_1} b\right).$$

Theorem 10. *Let w_1, w_2, w_3 be any positive integers, such that r does not divide w_1, w_2, w_3. Then we have the following two symmetries in symmetries in w_1, w_2, w_3:*

$$\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_3}}(w_1 y) B_{l,\chi,\xi^{w_1}}(w_2 y) B_{m,\chi,\xi^{w_2}}(w_3 y) w_3^k w_1^l w_2^m$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_2}}(w_1 y) B_{l,\chi,\xi^{w_1}}(w_3 y) B_{m,\chi,\xi^{w_3}}(w_2 y) w_2^k w_1^l w_3^m.$$

$$\tag{55}$$

Theorem 11. *Let w_1, w_2, w_3 be any positive integers, such that r does not divide $w_2 w_3, w_1 w_3, w_1 w_2$. Then we have the following two symmetries in w_1, w_2, w_3:*

$$\sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_1 - 1; \chi, \xi^{w_3}) S_l(dw_2 - 1; \chi, \xi^{w_1}) S_m(dw_3 - 1; \chi, \xi^{w_2}) w_3^k w_1^l w_2^m \tag{56}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_1 - 1; \chi, \xi^{w_2}) S_l(dw_3 - 1; \chi, \xi^{w_1}) S_m(dw_2 - 1; \chi, \xi^{w_3}) w_2^k w_1^l w_3^m. \tag{57}$$

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